# Recap

Tuesday, February 9, 2016 5:47 PM

Quantum fluid model: charged bosons coupled to EM field $\quad - \frac{1}{ng^*} \nabla\left(\frac{\Lambda}{2}\vec{J}^2\right)$

$$i\hbar\partial_t \psi = \left\{ \frac{1}{2m}\left[-i\hbar\vec{\nabla} - g\vec{A}\right]^2 + gU \right\}\psi$$

↳ Wavefunction: $\psi \sim \sqrt{n}\, e^{i\theta}$, $n$: charge carrier dty

↳ 1st London equation $\quad \frac{\partial}{\partial t}(\Lambda\vec{J}) = \vec{E} - \frac{1}{ng}\vec{\nabla}\left(\frac{\Lambda}{2}\vec{J}^2\right)$

↳ $\Lambda = \frac{m}{g^2 n}$ ~ inductance $\quad \sqrt{\frac{\mu}{\Lambda}}$ ~ penetration depth

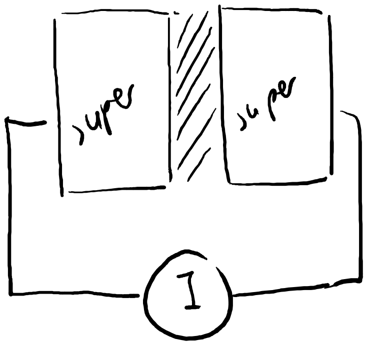
↳ Charge current $\quad \vec{J} \sim gn\left[\frac{\hbar}{m}\vec{\nabla}\theta - \frac{g}{m}\vec{A}\right]$

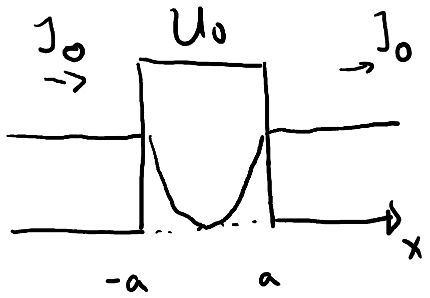
↳ Fluxoid quant. $\quad \oint_c (\Lambda\vec{J})\cdot d\vec{l} + \underbrace{\oint_c \vec{B}\cdot d\vec{s}}_{\Phi_{ext}} \in \Phi_0 \times \mathbb{Z} \qquad \Phi_0 = \frac{h}{g} = \frac{h}{2e}$ flux quantum

↳ Flux-phase $\quad \partial_t\theta = -\frac{gU}{\hbar}$

# Josephson relation

Saturday, February 6, 2016 6:28 PM




* We consider the superconducting current that can be established between two superconductors through a thin insulator
* We model this as a quantum tunnel process through a potential barrier for the Cooper pairs, which come with energy $E \ll U_0$
* We seek stationary solutions in absence of flux. We start w. $\bar{A}=0$

$$E\psi = \left[\frac{1}{2m'}\left(-i\hbar\nabla - q'\cancel{A}\right)^2 + V_0\right]\psi$$

* Outside the barrier we assume a constant dty and uniform current

$$\Lambda J_0 \sim \frac{\hbar}{q}\frac{d\theta}{dx} \rightarrow \begin{array}{l} \psi(x<-a) \sim \sqrt{n_L}\exp(i\theta_L(x)) \\ \psi(x>+a) \sim \sqrt{n_R}\exp(i\theta_R(x)) \end{array}$$

* Between boundaries

$$\frac{\hbar^2}{2m^*}\frac{d^2\psi}{dx^2} = (U_0 - E)\psi \rightarrow \psi(x) = C_1\cosh(x/\xi) + C_2\sinh(x/\xi) \qquad \xi = \sqrt{\frac{\hbar^2}{2m(U_0 - E_0)}}$$

property of insulator, not penetrat. depth

# Josephson relation (2)

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Matching boundary conditions $\psi(-a) = \sqrt{n_L}\, e^{i\theta_L}$, $\psi(+a) = \sqrt{n_R}\, e^{i\theta_R}$

$$C_1 = \frac{\sqrt{n_L}\, e^{i\theta_L} + \sqrt{n_R}\, e^{i\theta_R}}{2\cosh(a/\xi)}, \quad C_2 = \frac{\sqrt{n_R}\, e^{i\theta_R} - \sqrt{n_L}\, e^{i\theta_L}}{2\sinh(a/\xi)}$$

Notice that by defining $J_c$ with $(-q^*)$, we invert the order of phases *

$$J_s = \frac{q^*}{m^*} \mathrm{Re}\left(-\psi^* i\hbar \nabla \psi\right) = \frac{q^* \hbar}{m^* \xi} \mathrm{Im}(C_1^* C_2) = J_c \sin(\theta_L - \theta_R)$$ uniform along junction

$$J_c = \overset{*}{-} \frac{q^* \hbar}{2 m^* \xi} \frac{\sqrt{n_L n_R}}{\sinh(2a/\xi)} > 0$$ is the critical current below which tunnelling is sust.

More generally, provided $\vec{\nabla} \times \vec{A} = \vec{B} = 0$, we can write

$\vec{J} = J_c \sin\varphi$ with the gauge invariant phase

$$\varphi = \theta_L - \theta_R - \frac{2\pi}{\Phi_0} \int_L^R \vec{A} \cdot d\vec{l}$$

# Josephson relation (3)

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* Phase - voltage relations $\dfrac{\partial}{\partial t}\varphi = \dfrac{\partial}{\partial t}\theta_L - \dfrac{\partial}{\partial t}\theta_R - \dfrac{2\pi}{\Phi_0}\displaystyle\int_L^R \dfrac{\partial \bar{A}}{\partial t}\cdot d\bar{l}$ $\qquad \Phi_0 = \dfrac{|q^*|}{h}$

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n}\left(\cancel{J^2(-a) - J^2(+a)}\right)\right) - \frac{1}{\hbar}\left[q^* v(-a) - q^* v(+a)\right] - \frac{2\pi}{\Phi_0}\int_L^R \frac{\partial \bar{A}}{\partial t}\cdot d\bar{l}$$

$$= \int_L^R \left[\frac{q^*}{\hbar}\vec{\nabla} v - \frac{2\pi}{\Phi_0}\frac{\partial \vec{A}}{\partial t}\right] d\vec{l} = \frac{2\pi}{\Phi_0}\int_L^R \left[-\vec{\nabla} \underset{\uparrow}{v} - \frac{\partial \vec{A}}{\partial t}\right] d\vec{l} = \frac{2\pi}{\Phi_0}\underbrace{\int_L^R \vec{E}\cdot d\vec{l}}_{\text{voltage difference}}$$

electric potential

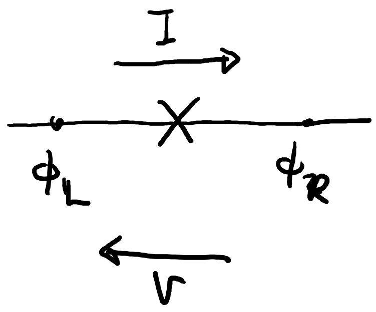
voltage difference $V_L - V_R$

* Circuit relations

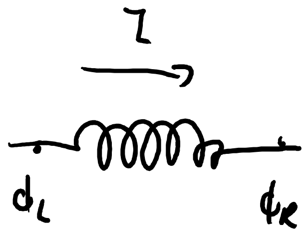
$$I = \int_{\text{cross sect}} \vec{J}\cdot d\vec{S} \sim I_c \sin[\varphi(t)]$$

$$\frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_0}\cdot V \sim \frac{2\pi}{\Phi_0}\cdot\frac{\partial}{\partial t}(\phi_L - \phi_R)$$

$$I = I_c \sin\left(\frac{2\pi}{\Phi_0}\cdot(\phi_L - \phi_R)\right)$$




similar to




$$\phi_L - \phi_R \sim \frac{1}{L} I$$

# Fluxoid quantization (2)

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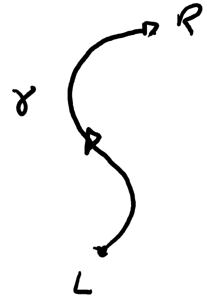
Remember $\frac{\hbar}{q^*} = \frac{\hbar}{-2e} = \frac{-\Phi_0}{2\pi}$

$$\oint_C (\Lambda \vec{J})\cdot d\vec{l} = \oint_C \frac{\hbar}{q^*}\nabla\theta\cdot d\vec{l} - \oint_C \vec{A}\cdot d\vec{l}$$




$\gamma$ is a directed path between two points

We take "L" to be an origin for the phase

$$\underbrace{\oint_\gamma (\Lambda\vec{J})\, d\vec{l} \times \left(\frac{-2\pi}{\Phi_0}\right)} = + \int_\gamma \nabla\theta\cdot d\vec{l} + \frac{2\pi}{\Phi_0}\int_\gamma \vec{A}\cdot d\vec{l}$$

$$\int_\gamma \nabla\varphi \equiv + \oint_\gamma \nabla\theta\cdot d\vec{l} - \frac{2\pi}{\Phi_0}\oint_{-\gamma}\vec{A}\cdot d\vec{l}$$

⇐ Definition of gauge invariant phase

closed paths

$$\frac{\Phi_0}{2\pi}\oint_C \bar{\nabla}\varphi\, d\bar{l} = \Phi_0 \times \mathbb{Z} + \int_{In(C)} \bar{B}\cdot d\bar{s} = \Phi_{ext}$$

Through phase-flux relation

$$\phi = \frac{\Phi_0}{2\pi}\cdot\varphi$$

$$\oint_C \bar{\nabla}\phi\cdot d\bar{l} = \Phi_{ext} = \Phi_0\cdot m \quad (m\in\mathbb{Z})$$

# Conventions

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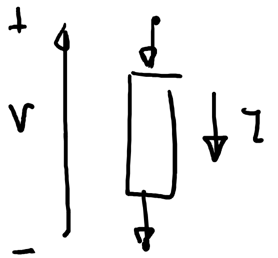
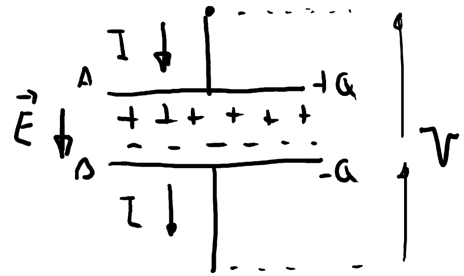
* Current is regarded in terms of positive charges.
* Such carriers flow from large to small electric potential




* On a capacitor, current flows lead to charge accumulation and establishment of a potential difference

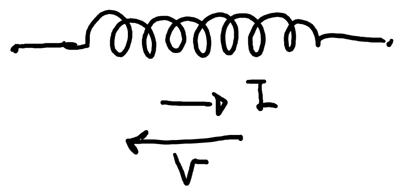



* Note that the sense of the flow that charges the capacitor is opposite to that of the established $V$

$$I = \frac{dQ}{dt} = C\dot{V}$$

* Note also that $V = \int_A^B E\,dx$

* On an inductor, a growth of current induces a potential that opposes the current (Lenz's law)




$$V = L\frac{dI}{dt}$$

# Applying the convention

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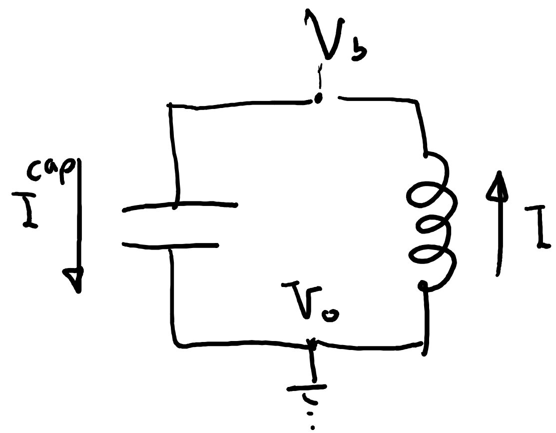
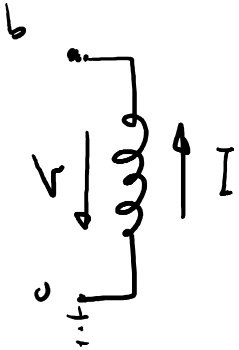


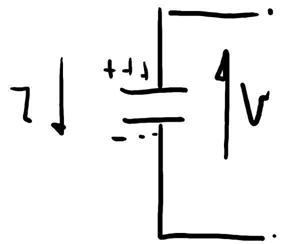

* The sum of the currents going into capacitive elements must be equal to the sum of currents coming from inductive elements

$$I^{cap}_{\vec{bo}} = I^{ind}_{\vec{ob}}$$




$$V_0 - V_b = L\frac{dI}{dt}$$

$$\int (V_0 - V_b)\,dt = LI$$




$$(V_b - V_0)\cdot C = Q$$

$$(\dot{V}_b - \dot{V}_0)C = I$$

↳ flux variables:

$$\frac{\phi_0 - \phi_b}{L} = (\ddot{\phi}_b - \ddot{\phi}_0)C \underset{\substack{\phi_0 = 0 \\ \text{grounded}}}{\Longrightarrow} \ddot{\phi}_b = -\frac{1}{LC}\phi_b$$

# Detailed rules

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1) Draw the graph for your circuit and assing nodes to the intersections or connections

2) Define one <u>flux variable</u> for each <u>node</u>

3) Draw a <u>directed</u> subgraph (tree) starting from one node and running <u>over all</u>

4) Define <u>branch fluxes</u> as the flux difference between consecutive nodes on that tree

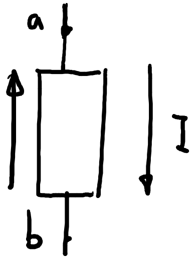
5) Assign <u>branch fluxes</u> to the remaining branches that close loops using fluxoid quantization

$$\sum_{C} \delta\phi = \Phi_{ext} \quad (+ 2\pi m \Phi_0 \approx 0)$$

6) Write differential equations for each node

$$\sum \text{currents} = 0$$

$\delta\phi = \phi_a - \phi_b$




1. Capacitor $I = C\,\delta\ddot{\phi}$
2. Inductor $I = \frac{1}{L}\,\delta\phi$
3. Junction $I = \frac{\Phi_0}{L_J}\sin(\delta\phi)$

7) Find lagrangian and quantize

# Driven capacitor

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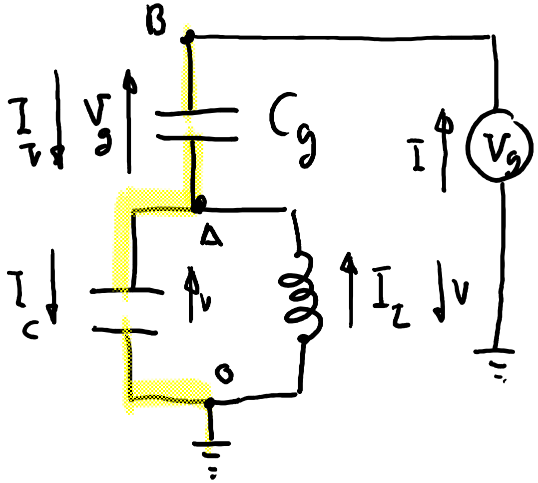


$$I_c - I_L = I_V$$

$$I_c = C\,(\ddot{\phi}_A - \ddot{\phi}_0)$$

$$I_V = C_g\,(\ddot{\phi}_B - \ddot{\phi}_A)$$

$$I_L = \frac{1}{L}\,(\phi_0 - \phi_A)$$

We assume

1) No trapped fluxes
2) Grounded circuit $\phi_0 = 0$
3) Constant voltage due to source $V_g \Rightarrow \dot{\phi}_B = V_g$

$$C\,\ddot{\phi}_A + \frac{1}{L}\,\phi_A = C_g\,(\dot{V}_g - \ddot{\phi}_A)$$

$$\frac{d}{dt}\left[(C+C_g)\left(\dot{\phi}_A - \frac{C_g}{C+C_g}V_g\right)\right] = -\frac{1}{L}\,\phi_A$$

$$\mathcal{L} = \frac{1}{2}\,C_\Sigma\left(\dot{\phi}_A - \frac{C_g}{C_\Sigma}V_g\right)^2 - \frac{1}{2L}\,\phi_A^2$$

$$q = C_\Sigma\,\dot{\phi}_A - C_g V_g$$

$$H = \frac{1}{2C_\Sigma}\Big(q + \underbrace{\frac{C_g}{C_\Sigma}V_g}_{q_g}\Big)^2 + \frac{1}{2L}\,\phi_A^2$$

# Driven capacitor (2)

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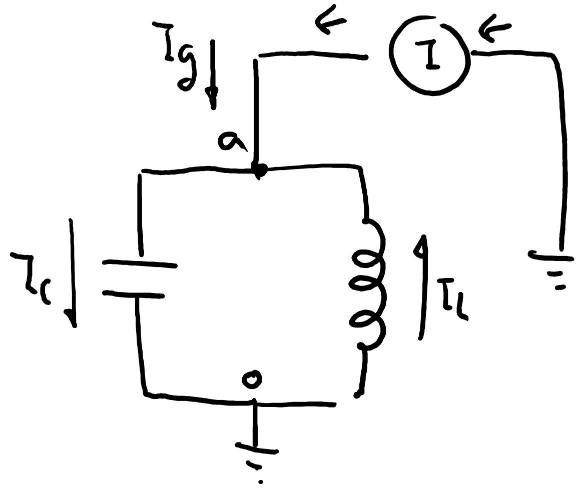


$$\left.\begin{aligned} I_C - I_L &= I_g \\ I_C &= C\left(\ddot{\phi}_A - \ddot{\phi}_0\right) \\ I_L &= \frac{1}{L}\left(\phi_0 - \phi_A\right) \end{aligned}\right\} \quad C\ddot{\phi}_A = -\frac{1}{L}\phi_A + I_g$$

$$\mathcal{L} = \frac{1}{2}C\left(\dot{\phi}_A\right)^2 - \frac{1}{2L}\left(\phi_A - LI_g\right)^2 + \text{constants}$$

$$H = \frac{1}{2C}q^2 + \frac{1}{2L}\left(\phi_A - LI_g\right)^2$$